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# POSITIVE SOLUTIONS OF NONLINEAR FRACTIONAL THREE-POINT BOUNDARY-VALUE PROBLEM

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In this paper, we study the existence of positive solutions to the boundary-value problem with fractional order

$$({}_a^C D^\alpha y)(t) + q(t) f(y) = 0, \quad 0 \leq a < t < b, \quad 1 < \alpha < 2,$$

$$y(a) = 0, \quad y(b) = \beta y(\eta),$$

where $a < \eta < b$ and $\beta(\eta - a) - b + a \neq 0$. We prove the existence of at least one positive solution when $f$ is either superlinear or sublinear using the well-known Guo-Lakshmikantham fixed point theorem in cones. Moreover, the convexity and concavity of the solutions are investigated with respect to the behavior of the function $q$.

## 1. Introduction

In the last decades, the investigation of multi-point boundary value problem for linear second order ordinary differential equations was begun by Il'in and Moiseev [10, 11]. The study of three-point BVPs for nonlinear integer-order ordinary differential equations was initiated by Gupta [7]. Many authors since

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To the memory of Prof. Maryam Mirzakhani then considered the existence and multiplicity of solutions (or positive solutions) of three-point BVPs for nonlinear integer-order ordinary differential equations. To identify a few, we refer the reader to [15, 16, 24] and the references therein.

In 2000, using the fixed point index theorems, Leray-Schauder degree and upper and lower solutions, Ma [15] investigated the following second-order three-point boundary value problem

$$\begin{aligned} u'' + \lambda h(t) f(u) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad cu(\eta) = u(1), \end{aligned} \tag{1}$$

where

**(A)** $\lambda$ is a positive parameter; $\eta \in (0,1)$ and $0 < c\eta < 1$;

**(B)** $h : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subset of positive measure;

**(C)** $f : [0,\infty) \to [0,\infty)$ is continuous;

**(D)** $f_\infty := \lim_{u\to\infty} \dfrac{f(u)}{u} = \infty.$

In the result of He and Ge [8], utilizing Leggett-Williams fixed-point theorem [13], the multiplicity of positive solutions of the following problem has been concerned:

$$\begin{aligned} u'' + f(t,u) = 0, \ t \in (0,1), \\ u(0) = 0, \quad cu(\eta) = u(1), \end{aligned}$$

where $0 < \eta < 1$, $c > 0$ and $0 < c\eta < 1$. Moreover, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $f(t,\cdot)$ does not vanish identically on any subset of $[0,1]$ with positive measure.

In the last decades, fractional calculus and fractional differential equations have attracted much attention, we refer for instance to [1, 2, 14, 18, 19, 26] and references therein. It is found that many phenomena can be modeled with the aid of fractional derivatives or integrals, such as fractional Brownian motion [3], anomalous diffusion [9, 17], etc. This motivates us to remodel the problem (1) by a fractional order and study on it.

Throughout this paper, we consider the existence of positive solutions to the three-point boundary value problem consisting by the fractional differential equation

$$({}^C_aD^\alpha y)(t) + q(t) f(y) = 0, \quad 0 \le a < t < b, \tag{2}$$

where ${}_a^C D^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha < 2$, subject to the boundary conditions

$$y(a) = 0, \quad y(b) = \beta y(\eta), \quad a < \eta < b, \tag{3}$$

where $f, q$ satisfy

**(H1)** $f \in C([0,\infty),[0,\infty))$;

**(H2)** $q \in C([a,b],[0,\infty))$.

By taking

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u},$$

we set $f_0 = 0$ and $f_\infty = \infty$ corresponding to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ corresponding to the sublinear case. Here, in this paper, our goal is to present some existence results for positive solutions to (2)-(3), assuming that $f$ is either superlinear or sublinear. The technique of proof of our main result is based upon the well-known Guo-Lakshmikantham fixed point theorem [6] in a cone.

**Theorem 1.1.** *[6] Let E be a Banach space, and let $K \subseteq E$ be a cone. Assume $\Omega_1, \Omega_2$ are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

*be a completely continuous operator such that:*

**(i)** $\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_2$; *or*

**(ii)** $\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_2$.

*Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

## 2. Preliminaries and auxiliary facts

For completeness, in this section, we gather some fundamental definitions of Caputo's derivatives of fractional order which can be found in ([12], [20], [21]) together with some simple crucial lemmas which will be needed further on.

**Definition 2.1.** Let $\alpha \ge 0$ and $f$ be a real function defined on $[a,b]$. The *Riemann-Liouville fractional integral* of order $\alpha$ for a continuous function $f : (a,\infty) \to \mathbb{R}$ is defined by $({}_aI^0 f)(x) = f(x)$ and

$$({}_aI^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t \in [a,b],$$

where $\Gamma(\cdot)$ is the Gamma function.

**Definition 2.2.** For a continuous function $f:(a,\infty)\to\mathbb{R}$ the Riemann-Liouville fractional derivative of fractional order $\alpha>0$ is defined by

$$^{RL}D^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^{n}\int_a^t (t-s)^{n-\alpha-1}f(s)ds,\ \ n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number $\alpha$.

For $\alpha<0$, we use the convention that $D^{\alpha}y=I^{-\alpha}y$. Also for $\beta\in[0,\alpha)$, it is valid that $D^{\beta}I^{\alpha}y=I^{\alpha-\beta}y$.

**Definition 2.3.** The *Caputo fractional derivative* of order $\alpha\geq 0$ is given by $({}^{C}_{a}D^{0}f)(t)=f(t)$ and $({}^{C}_{a}D^{\alpha}f)(t)=({}_{a}I^{m-\alpha}D^{m}f)(t)$ for $\alpha>0$, where $m$ is the smallest integer greater or equal to $\alpha$. Besides, it can be formulated by

$$^{C}D^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds,\ \ n=[\alpha]+1,\quad f\in AC^{n}([a,b]),$$

where $\alpha\notin\mathbb{N}_0$ and $AC^{n}([a,b])$ represents the space of all absolutely continuous functions having absolutely continuous derivative up to $(n-1)$ (see also [12]).

The Green function for the BVP (2)-(3) can be obtained by using an important lemma derived by Zhang [25] as follows:

**Lemma 2.4.** *Let $\alpha>0$, then in $C(0,T)\cap L(0,T)$ the differential equation*

$$^{C}D^{\alpha}_{0+}u(t)=0$$

*has solutions $u(t)=c_0+c_1t+c_2t^2+\cdots+c_nt^{n-1}$, $c_i\in\mathbb{R}$, $i=0,1,\cdots,n$, $n=[\alpha]+1$.*

Moreover, it has been proved that $I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_nt^{n-1}$ for some $c_i\in\mathbb{R}$, $i=0,1,\cdots,n$, $n=[\alpha]+1$ (see Lemma 2.3 in [25]).

In the following we present a pivotal lemma which will play major role in our next analysis and concern a linear variant of problem (2)-(3).

**Lemma 2.5.** *For $g\in C([a,b],[0,\infty))$, the problem*

$$({}^{C}_{a}D^{\alpha}y)(t)+g(t)=0,\qquad 0\leq a<t<b, \tag{4}$$

*with order $1<\alpha<2$ and the boundary condition* (3) *has a unique solution*

$$\begin{aligned}y(t)=&-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds\\&+\frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\Big(\int_a^b (b-s)^{\alpha-1}g(s)ds-\beta\int_a^{\eta}(\eta-s)^{\alpha-1}g(s)ds\Big).\end{aligned}$$

*Proof.* Applying the Riemann-Liouville fractional integral ${}_aI^{\alpha}$ for (4)-(3) and the imposed boundary conditions together with a fact from fractional calculus theory we see that $y \in C[a,b]$ is a solution of (4)-(3) if and only if

$$y(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds \tag{5}$$

for some real constants $c_0$ and $c_1$ (see Lemma 2.4). Since $y(a) = 0$ we get immediately that $c_0 = 0$. Now,

$$\begin{aligned} y(b) = \beta y(\eta) \;\; &\Leftrightarrow \;\; c_1(b-a) - \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} g(s)ds \\ &= c_1\beta(\eta-a) - \frac{\beta}{\Gamma(\alpha)} \int_a^{\eta} (\eta-s)^{\alpha-1} g(s)ds \\ &\Leftrightarrow \;\; c_1 = \frac{1}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \left( \int_a^b (b-s)^{\alpha-1} g(s)ds \right. \\ &\qquad \left. -\beta \int_a^{\eta} (\eta-s)^{\alpha-1} g(s)ds \right). \end{aligned}$$

Hence, equality (5) becomes

$$\begin{aligned} y(t) \;\; = \;\; & -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds + \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \times \\ & \times \left( \int_a^b (b-s)^{\alpha-1} g(s)ds - \beta \int_a^{\eta} (\eta-s)^{\alpha-1} g(s)ds \right). \end{aligned}$$

□

**Lemma 2.6.** *Suppose that $g \in C^2([a,b];\mathbb{R})$ and $g(a) \geq 0$.*

**(a)** *If g is convex, then the unique solution of* (4)-(3) *is concave.*

**(b)** *If g is concave, then the unique solution of* (4)-(3) *is convex.*

*Proof.* In order to prove the validity of (a), first, by the definition of the Caputo's derivative, it is easily seen from (4)-(3) that

$$I_a^{2-\alpha}(y''(t)) = -g(t).$$

Then it follows that

$$I_a^{\alpha}(I_a^{2-\alpha}(y''(t))) = -I_a^{\alpha}(g(t)).$$

That is,

$$I_a^2(y''(t)) = -I_a^{\alpha}(g(t)).$$

Hence, we can obtain

$$y''(t) = -\frac{d^2}{dt^2} I_a^\alpha(g(t)) = -I_a^\alpha(g''(t)) = -{}^{RL}D_a^{2-\alpha} g(t).$$

On the other hand, from the fractional calculus we know that

$${}^{C}D_a^\alpha g(t) = {}^{RL}D_a^\alpha g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \quad (n = \lfloor \alpha \rfloor + 1),$$

see also [12]. Since $0 < 2-\alpha < 1$ then we get

$${}^{C}D_a^{2-\alpha} g(t) = {}^{RL}D_a^{2-\alpha} g(t) - \frac{g(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}, \qquad a < t \leq b,$$

which implies that

$$\begin{aligned} y''(t) &= -\left({}^{C}D_a^{2-\alpha} g(t) + \frac{g(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\right) \\ &= -\left(I_a^\alpha(g''(t)) + \frac{g(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\right) \end{aligned}$$

which is obviously non-positive for all $t \in (a,b]$ and so the solution of (4)-(3) is concave. The proof of the second part is quite similar. □

**Lemma 2.7.** *Let* $0 < \beta\eta < b$ *and* $g \in C^2([a,b];\mathbb{R})$ *be a convex function with* $g(a) \geq 0$. *Then the unique solution of the problem* (4)-(3) *satisfies* $y(t) \geq 0$ *for all* $t \in [a,b]$ *and is concave.*

*Proof.* Following Lemma 2.6 we see that $y(t)$ is concave down on $(a,b)$. If $y(b) \geq 0$, then the concavity of $y$ and the boundary condition $y(a) = 0$ yield $y(t) \geq 0$ for all $t \in [a,b]$. Otherwise, letting $y(b) < 0$, we have $y(\eta) < 0$ and

$$y(b) = \beta y(\eta) > \frac{b}{\eta} y(\eta),$$

which contradicts the concavity of $y$ and the proof is complete. □

**Proposition 2.8.** *Suppose that* $\beta\eta > b$ *and* $g \in C^2([a,b];\mathbb{R})$ *is a convex function with* $g(a) \geq 0$. *Then the problem* (4)*-(3) has no positive solution.*

*Proof.* Suppose the contrary, (4)-(3) has a positive solution $y$. If $y(b) > 0$, then $y(\eta) > 0$ and

$$y(b) = \beta y(\eta) > \frac{b}{\eta} y(\eta),$$

which contradicts the concavity of $y$, since $g$ is convex. Now, let $y(b) = 0$ and $y(r) > 0$ for some $r \in (a,b)$, then

$$y(\eta) = y(b) = 0, \qquad \eta \neq r.$$

This together with the condition $y(a) = 0$ implies that $y$ is not concave. Indeed,

$$\begin{aligned} &r \in (a,\eta) \Longrightarrow y(r) < y(\eta) = y(b) = 0 \\ &r \in (\eta,b) \Longrightarrow y(r) > y(\eta) = y(b) = 0 \Longrightarrow y(a) < 0 \end{aligned}$$

which both cases show a contradiction using the concavity of $y$. □

**Lemma 2.9.** *Let $0 < \beta\eta < b$, $\beta(a-\eta)+b-a \neq 0$, and $g \in C^2([a,b];\mathbb{R})$ be a convex function with $g(a) \geq 0$. Then the solution of Eq.* (4)-(3) *satisfies*

$$\min_{t\in[\eta,b]} y(t) \geq \gamma\|y\|$$

*where*

$$\gamma = \min\left\{\frac{\beta(b-\eta)}{\beta(a-\eta)+b-a}, \frac{\beta\eta}{b}, \frac{\eta}{b}\right\}. \tag{6}$$

*Proof.* We split the proof into the following cases.

**Case 1.** We encounter with the case $0 < \beta < 1$. Following Lemma 2.7 and initial conditions we know that $y(\eta) \geq y(b)$. Now, let $y(\hat{t}) = \|y\|$ for some $\hat{t} \in (a,b]$. Assume that $\hat{t} \leq \eta < b$, then

$$\min_{t\in[\eta,b]} y(t) = y(b). \tag{7}$$

On the other hand, from the concavity of the solution $y$ we see

$$\frac{y(\eta)-y(\hat{t})}{\eta-\hat{t}} \geq \frac{y(b)-y(\eta)}{b-\eta}$$

which shows that

$$\begin{aligned} y(\hat{t}) &\leq \frac{(b-\eta)+(1-\beta)(\eta-\hat{t})}{\beta(b-\eta)}y(b) \\ &\leq \frac{(b-\eta)+(1-\beta)(\eta-a)}{\beta(b-\eta)}y(b) \\ &= \frac{\beta(a-\eta)+b-a}{\beta(b-\eta)}y(b). \end{aligned}$$

This together with (7) yields that

$$\min_{t\in[\eta,b]} y(t) \geq \frac{\beta(b-\eta)}{\beta(a-\eta)+b-a}\|y\|.$$

Now, let us take $\eta < \hat{t} < b$, then

$$\min_{t\in[\eta,b]} y(t) = y(b). \tag{8}$$

Using the concavity of $y$ we conclude

$$\frac{y(\eta)}{\eta} \geq \frac{y(\hat{t})}{\hat{t}}.$$

This together with the boundary condition $y(b) = \beta y(\eta)$ implies that

$$\frac{y(b)}{\beta\eta} \geq \frac{y(\hat{t})}{\hat{t}} > \frac{1}{b}\|y\|$$

which means

$$\min_{t\in[\eta,b]} y(t) > \frac{\beta\eta}{b}\|y\|.$$

**Case 2.** Suppose that $1 \leq \beta < \frac{b}{\eta}$. Then we have $y(\eta) \leq y(b)$. Now, by setting $y(\hat{t}) = \|y\|$ we see that $\eta \leq \hat{t} \leq b$. We notice that if $a < \hat{t} < \eta$, then the point $P_\eta = (\eta, y(\eta))$ is below the straight line given by the points $P_b = (b, y(b))$ and $P_{\hat{t}} = (\hat{t}, y(\hat{t}))$ and this contradicts the concavity of $y$. The recent facts guarantee the following equality:

$$\min_{t\in[\eta,b]} y(t) = y(\eta).$$

Similar to the former case and using Lemma 2.7 we obtain

$$\frac{y(\eta)}{\eta} \geq \frac{y(\hat{t})}{\hat{t}}$$

which implies

$$\min_{t\in[\eta,b]} y(t) \geq \frac{\eta}{b}\|y\|$$

and the consequence follows. □

## 3. Main result

Based on the lemmas presented in previous section we derive our main result as follows.

**Theorem 3.1.** *Assume that (H1) and (H2) hold. Then the problem (2)-(3) has at least one positive solution in the case*

**(i)** $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*

**(ii)** $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

*Proof.* Let us first consider the case (i):
**Superlinear case.** Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We want to establish the existence of a positive solution of (2)-(3). Following the proof of Lemma 2.5, problem (2)-(3) has a solution $y = y(t)$ if and only if $y$ solves the operator equation

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)f(y(s))ds + \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\times \\ &\times\left(\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds - \beta\int_a^\eta (\eta-s)^{\alpha-1}q(s)f(y(s))ds\right) \\ &\overset{\text{def}}{=} Ay(t). \end{aligned} \tag{9}$$

Set

$$K := \{y \mid y \in C[a,b],\ y \geq 0,\ \min_{\eta\leq t\leq b} y(t) \geq \gamma\|y\|\}, \tag{10}$$

where $\gamma$ is given by (6). It is clear that $K$ is a cone in $C[a,b]$. Moreover, by Lemma 2.9, $AK \subset K$. It is also easy to see that $A : K \to K$ is completely continuous.

Now since $f_0 = 0$, we may take $r_1 > 0$ such that $f(y) \leq \varepsilon y$, for $0 < y < r_1$, where $\varepsilon > 0$ satisfies

$$\frac{\varepsilon(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)ds\right) < 1. \tag{11}$$

Hence, if $y \in K$ and $\|y\| = r_1$, then following (9) and (11), we derive

$$\begin{aligned} Ay(t) &\leq \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)f(y(s))ds\right) \\ &\leq \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)\varepsilon y(s)ds\right) \\ &\leq \frac{\varepsilon(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)\|y\|ds\right) \\ &= \frac{\varepsilon r_1(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1}q(s)ds\right). \end{aligned} \tag{12}$$

Now if we set

$$\Omega_1 = \{y \in C[a,b] \ | \ \|y\| < r_1\}, \tag{13}$$

then (12) yields that $\|Ay\| \leq \|y\|$, for all $y \in K \cap \partial\Omega_1$. Moreover, since $f_\infty = \infty$, there exists $\hat{r}_2 > 0$ so that $f(u) \geq \rho u$ for all $u \geq \hat{r}_2$ where $\rho > 0$ is taken so that

$$\frac{\rho\gamma(\eta - a)}{\Gamma(\alpha)(b - a - \beta(\eta - a))} \int_\eta^b (b-s)^{\alpha-1} q(s)ds \geq 1. \tag{14}$$

Suppose $r_2 = \max\{2r_1, \hat{r}_2\gamma^{-1}\}$ and $\Omega_2 = \{y \in C[a,b] \ | \ \|y\| < r_2\}$, then $y \in K$ with $\|y\| = r_2$ yields

$$\min_{\eta \leq t \leq b} y(t) \geq \gamma\|y\| \geq \hat{r}_2,$$

and hence

$$\begin{aligned}
Ay(\eta) &= -\frac{1}{\Gamma(\alpha)} \int_a^\eta (\eta - s)^{\alpha-1} q(s) f(y(s))ds \\
&\quad + \frac{\eta - a}{\Gamma(\alpha)(b - a - \beta(\eta - a))} \left( \int_a^b (b-s)^{\alpha-1} q(s) f(y(s))ds \right. \\
&\quad \left. - \beta \int_a^\eta (\eta - s)^{\alpha-1} q(s) f(y(s))ds \right) \\
&= -\frac{1}{\Gamma(\alpha)(b - a - \beta(\eta - a))} \left( (b-a) \int_a^\eta (\eta - s)^{\alpha-1} q(s) f(y(s))ds \right. \\
&\quad \left. - (\eta - a) \int_a^b (b-s)^{\alpha-1} q(s) f(y(s))ds \right) \\
&= -\frac{1}{\Gamma(\alpha)(b - a - \beta(\eta - a))} \left( \int_a^\eta \left[ (b-a)(\eta - s)^{\alpha-1} \right. \right. \\
&\quad \left. - (\eta - a)(b-s)^{\alpha-1} \right] q(s) f(y(s))ds \\
&\quad \left. - (\eta - a) \int_\eta^b (b-s)^{\alpha-1} q(s) f(y(s))ds \right).
\end{aligned} \tag{15}$$

On the other hand, by the fact that

$$0 \leq \frac{\eta - s}{b - s} \leq \left( \frac{\eta - s}{b - s} \right)^{\alpha - 1} \leq \frac{\eta - a}{b - a} < 1, \quad a \leq s \leq \eta < b, \quad 1 < \alpha < 2,$$

we see that

$$Ay(\eta) \geq \frac{\eta - a}{\Gamma(\alpha)(b - a - \beta(\eta - a))} \int_\eta^b (b-s)^{\alpha-1} q(s) f(y(s))ds.$$

Therefore, for $y \in K \cap \partial\Omega_2$,

$$\|Ay\| \geq \frac{\rho\gamma(\eta-a)\|y\|}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_\eta^b (b-s)^{\alpha-1} q(s)ds \geq \|y\|.$$

Consequently, by the first part of the Guo-Lakshmikantham fixed point theorem, it follows that $A$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $r_1 \leq \|y\| \leq r_2$. This finishes the proof of superlinear part of the theorem.

Now we consider the case (ii):

**Sublinear case.** Suppose then that $f_0 = \infty$ and $f_\infty = 0$. Let us first take $r_3 > 0$ such that $f(y) \geq \mu y$ for $0 < y < r_3$, where

$$\frac{\mu\gamma(\eta-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_\eta^b (b-s)^{\alpha-1} q(s)ds \geq 1. \tag{16}$$

Utilizing the same technique as used in (15), one can obtain that

$$\begin{aligned} Ay(\eta) = & -\frac{1}{\Gamma(\alpha)} \int_a^\eta (\eta-s)^{\alpha-1} q(s) f(y(s))ds \\ & + \frac{\eta-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \left( \int_a^b (b-s)^{\alpha-1} q(s) f(y(s))ds \right. \\ & \left. - \beta \int_a^\eta (\eta-s)^{\alpha-1} q(s) f(y(s))ds \right) \\ \geq & \frac{\eta-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_\eta^b (b-s)^{\alpha-1} q(s) f(y(s))ds \\ \geq & \frac{\mu(\eta-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_\eta^b (b-s)^{\alpha-1} q(s) y(s)ds. \end{aligned} \tag{17}$$

Therefore, we may set $\Omega_3 = \{y \in C[a,b] \mid \|y\| < r_3\}$ such that $\|Ay\| \geq \|y\|$ for $y \in K \cap \partial\Omega_3$.

On the other hand, since $f_\infty = 0$ then there is $\hat{r}_4 > 0$ such that $f(y) \leq \xi y$ for $y \geq \hat{r}_4$ where $\xi > 0$ enjoys

$$\frac{\xi(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \left( \int_a^b (b-s)^{\alpha-1} q(s)ds \right) \leq 1. \tag{18}$$

Now, we must consider two distinct cases as follows:

Case (I). Let us assume that $f$ is bounded, say $f(y) \leq M$ for all $y \geq 0$. For this case, we set

$$r_4 = \max\left\{ 2r_3, \frac{M(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))} \int_a^b (b-s)^{\alpha-1} q(s)ds \right\}$$

such that for $y \in K$ with $\|y\| = r_4$ we get

$$\begin{aligned}
Ay(t) &= -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} q(s) f(y(s))ds \\
&\quad + \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\left(\int_a^b (b-s)^{\alpha-1} q(s) f(y(s))ds \right. \\
&\quad \left. -\beta\int_a^\eta (\eta-s)^{\alpha-1} q(s) f(y(s))ds\right) \\
&\leq \frac{t-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_a^b (b-s)^{\alpha-1} q(s) f(y(s))ds \\
&\leq \frac{M(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_a^b (b-s)^{\alpha-1} q(s)ds \\
&\leq r_4
\end{aligned}$$

which yields $\|Ay\| \leq \|y\|$.

Case (II). Now, suppose that $f$ is unbounded, then we derive from (H1) that there exists $r_4$ such that

$$r_4 > \max\left\{2r_3, \frac{\hat{r}_4}{\gamma}\right\} \quad \text{s.t.} \quad f(y) \leq f(r_4) \quad \text{for} \quad 0 < y \leq r_4$$

and it would be possible because $f$ is unbounded. Using (18), for any $y \in K$ with $\|y\| = r_4$ we conclude that

$$\begin{aligned}
Ay(t) &\leq \frac{b-a}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_a^b (b-s)^{\alpha-1} q(s) f(r_4)ds \\
&\leq \frac{(b-a)}{\Gamma(\alpha)(b-a-\beta(\eta-a))}\int_a^b (b-s)^{\alpha-1} q(s)\xi r_4 ds \\
&\leq r_4.
\end{aligned}$$

Hence, in any case we may set

$$\Omega_4 = \{y \in C[a,b] \mid \|y\| < r_4\},$$

and then we may obtain $\|Ay\| \leq \|y\|$. Based on the second part of Guo-Lakshmikantham fixed point theorem, it follows that BVP (2)-(3) has a positive solution and the consequence follows. □

## 4. A concrete example

Concerning with the existence of positive solution of BVP (2)-(3), we now give an example to illustrate the effciency of our main result. Let us first recall some auxiliary facts as follows.

As we know, analytic solutions to fractional-order differential equations are often expressed in terms of the Mittag-Leffler function. The Mittag-Leffler function $E_{\alpha,\beta}$ is a special function, a complex function which relates to two complex parameters $\alpha$ and $\beta$ (it is also worth mentioning that it was firstly introduced as a one-parameter function). The Mittag-Leffler function is considered as a generalization of the exponential function. It may be given by the following series when the real part of $\alpha$ is strictly positive

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C},$$

which is of great importance for the fractional calculus. In the case $\alpha$ and $\beta$ are real and positive, the series converges for all values of the argument $z$, so the Mittag-Leffler function is an entire function.

**Example 4.1.** Consider the following boundary-value problem with fractional order

$$\begin{aligned} &({}_0^C D^{1.5} y)(t) + q(t)\sqrt[3]{y^2} = 0, \qquad 0 < t < 1, \\ &y(0) = 0, \qquad y(1) = \sqrt{3}y(0.5) \end{aligned} \tag{19}$$

where

$$q(t) = \frac{4\pi^{11/6} t^{5/6} \sum_{k=0}^{\infty} \frac{\left(\frac{-4\pi^2 t^2}{9}\right)^k}{(4k+3)!!}}{27\left(\frac{2}{9}\sum_{k=0}^{\infty} \frac{\left(\frac{-4\pi^2 t^2}{9}\right)^k}{(2k+2)!}\right)^{\frac{1}{3}}} < \infty, \qquad t \in (0,1)$$

and $n!!$ is called the double factorial and given by $n!! = n(n-2)(n-4)\cdots 5\cdot 3\cdot 1$ for odd $n > 0$ and $n!! = n(n-2)(n-4)\cdots 6\cdot 4\cdot 2$ for even $n > 0$.

First we note that $f(u) = \sqrt[3]{u^2}$ is a sublinear function. We claim that BVP (19) has a solution $y = \sin(\frac{\pi t}{3})$ which is concave on $[0,1]$. In order to prove it, bring in mind that

$$ {}_0^C D^{\alpha} \sin\lambda t = -\frac{1}{2} i(i\lambda)^n t^{n-\alpha}(E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t)) \tag{20}$$

such that $\lambda \in \mathbb{C}, \alpha \in \mathbb{R}, n \in \mathbb{N}$, and $n-1 < \alpha < n$. The formula as above also can be represented in the terms of the so-called hypergeometric functions (sometimes called the Kummer or confluent functions), see also [4]. Based on (20), we obtain

$$
\begin{aligned}
{}_0^C D^{\frac{3}{2}} \sin\frac{\pi t}{3} &= -\frac{1}{2} i(\frac{\pi}{3} i)^2 \sqrt{t}\left(E_{1,\frac{3}{2}}(\frac{i\pi t}{3}) - E_{1,\frac{3}{2}}(-\frac{i\pi t}{3})\right)\\
&= \frac{\pi^2 i}{18}\sqrt{t}\left(\sum_{k=0}^{\infty}\frac{(\frac{i\pi t}{3})^k}{\Gamma(k+\frac{3}{2})} - \sum_{k=0}^{\infty}\frac{(\frac{-i\pi t}{3})^k}{\Gamma(k+\frac{3}{2})}\right)\\
&= \frac{\pi^2}{18}\sqrt{t}\left(\sum_{k=0}^{\infty}\frac{i(\frac{i\pi t}{3})^k}{\Gamma(k+\frac{3}{2})}\left(1-(-1)^k\right)\right)\\
&= \frac{\pi^2}{18}\sqrt{t}\left(\sum_{k=1}^{\infty}\frac{i(\frac{i\pi t}{3})^{2k-1}}{\Gamma(2k-1+\frac{3}{2})}\left(1-(-1)^{2k-1}\right)\right)\\
&= \frac{\pi}{3\sqrt{t}}\left(\sum_{k=1}^{\infty}\frac{(-\frac{\pi^2 t^2}{9})^k}{\Gamma(2k+\frac{1}{2})}\right)\\
&= -\frac{\pi^3 t^{\frac{3}{2}}}{27}\left(\sum_{k=0}^{\infty}\frac{(-\frac{\pi^2 t^2}{9})^k}{\Gamma(2k+\frac{5}{2})}\right),
\end{aligned}
$$

making use of the fact that

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \qquad n = 0,1,2,3,\cdots,$$

we get the following formula

$${}_0^C D^{3/2} \sin\frac{\pi t}{3} = -\frac{4\pi^{5/2} t^{3/2}}{27}\left(\sum_{k=0}^{\infty}\frac{(-\frac{4\pi^2 t^2}{9})^k}{(4k+3)!!}\right) < \infty.$$

Moving forward, using the series expansion of cosine we derive

$$
\begin{aligned}
q(t)\sqrt[3]{\sin^2\frac{\pi t}{3}} &= \frac{4\pi^{11/6} t^{5/6} \sum_{k=0}^{\infty}\frac{(\frac{-4\pi^2 t^2}{9})^k}{(4k+3)!!}}{27\left(\frac{2}{9}\sum_{k=0}^{\infty}\frac{(\frac{-4\pi^2 t^2}{9})^k}{(2k+2)!}\right)^{\frac{1}{3}}} \times \sqrt[3]{\frac{2\pi^2 t^2}{9}}\left(\sum_{k=0}^{\infty}\frac{(-\frac{4\pi^2 t^2}{9})^k}{(2k+2)!}\right)^{\frac{1}{3}}\\
&= -{}_0^C D^{\frac{3}{2}} \sin\frac{\pi t}{3}
\end{aligned}
$$

which means $y = \sin\frac{\pi t}{3}$ is the solution of BVP (19).

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